

04-27-09

p -adic differential operators
on automorphic forms and applications

Ultimate Goal: p -adic interpolation of
special L -values. Construction of p -adic
 L -functions attached to families of automorphic forms.

Example: $\chi =$ Hecke character of a CM field.

$$\text{Then } L(0, \chi) = \sum_{\text{CM points}} E(z)$$

Eisenstein series, holomorphic with
algebraic coefficients

$$\text{So } L(0, \chi) \text{ algebraic}$$

$$L(m, \chi) = \sum_{\text{CM points}} E_m(z)$$

CM
points



Eisenstein series, but just C^∞ .

~~Thus~~ Thus can't say $L(m, \chi)$ algebraic.

But $E_m = D_m E$, D_m a
differential operator.

Main feature: Algebraicity properties of L -functions

comes from algebraic properties of Eisenstein at

CM points

Obstacle: Most Eisenstein series are not holomorphic.

Key tool: Differential operators (introduced by H. Maass)

C^∞ -differential operators: They act on

~~the~~ \mathbb{C} -vector valued C^∞ -functions on

certain symmetric spaces like the upper half plane.

Let's denote these operators by D_g .

Let f be ^{or a modular form} ~~holomorphic~~ with algebraic Fourier coefficients.
Thm: $\text{Tr}(\mathcal{D}_g f)(z) \in \overline{\mathbb{Q}}$, ~~if~~ if

z is a CM point.

Usefulness: Eisenstein series E holomorphic w/ $\overline{\mathbb{Q}}$ coefficients

\Downarrow

$\mathcal{D}_g E$ algebraic at CM points

Ex: Modular forms case:

$f: \mathcal{H} \rightarrow \mathbb{C}$ satisfying automorphy

condition $f(\gamma z) = (cz+d)^k f(z)$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

Define \mathcal{D}_k is a differential operator,

$$\mathcal{D}_k: \left\{ \begin{array}{l} \text{weight } k \\ \text{modular forms} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{weight } k+2 \\ \text{modular forms} \end{array} \right\}$$

by

$$\partial_k(f) := y^{-k} \left(\frac{\partial}{\partial z} y^k f \right)$$

In general:

- (ρ, V) a rep'n of ~~$GL_n(\mathbb{C}) \times GL_n(\mathbb{C})$~~

$$K := \{ (a, b) \in GL_n(\mathbb{C}) \times GL_n(\mathbb{C}) \mid \det(a) = \det(b) \}$$

- $f: \mathcal{H} \xrightarrow{cM_{n \times n}(\mathbb{C})} V$

automorphic form if

~~$$f(\gamma z) = (cz + d)^{-k} f(z)$$~~

$$f(\gamma z) = \rho(cz + d, c(z^t) + d) f(z)$$

Then define differential operators
($z - \bar{z}^t, z^t - \bar{z}$)

$$(D_g f)(z) = \rho \left(\begin{bmatrix} \square \\ \square \end{bmatrix} \right)^{-1} D \left[\rho \left(\begin{bmatrix} \square \\ \square \end{bmatrix} \right) f(z) \right]$$

matrix of differential operators

$$\mathcal{L}_g : \left\{ \begin{array}{l} \text{weight } g \\ \text{automorphic} \\ \text{form} \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{weight " } g \otimes \tau \text{ " } \\ \text{automorphic form} \end{array} \right\}$$

Obstacle : Not much insight into
 p -adic world.

Resolution of this obstacle : There's a p -adic

analogue of the D_g :

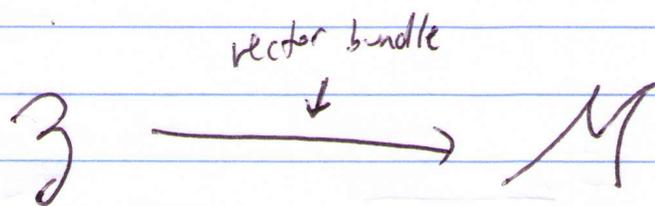
- dim 1 (N. Katz (1970's))
- dim 2 (Serre)
- higher dimensions (her thesis)

Perspective : Now : automorphic forms

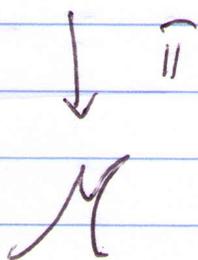
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sections of a vector bundle

on a moduli space \mathcal{M}
of abelian varieties
with PEL structure.



A = universal Abelian variety

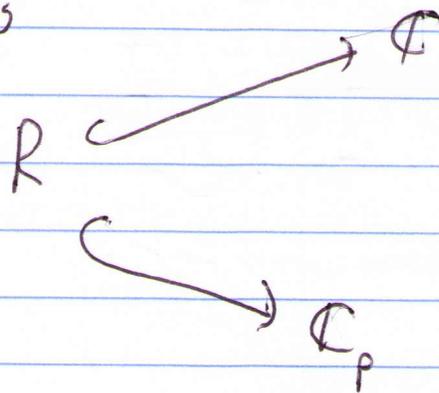


$$\omega = \pi_* \Omega_{A/\mathcal{M}}$$

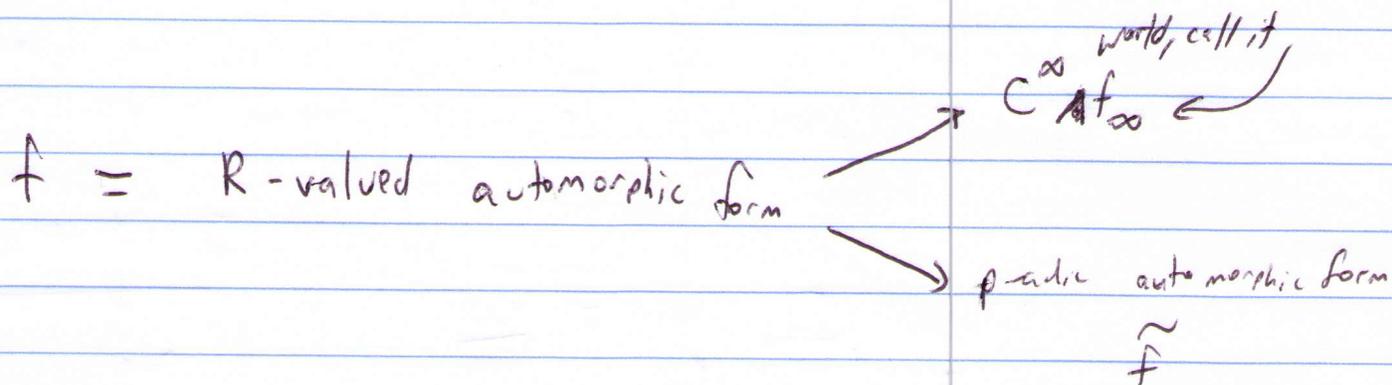


$R = \mathcal{O}_K$ -algebra, $K = \mathbb{C}M$ field.

Fix an embedding



$$R \hookrightarrow \varprojlim R/p^n R$$



Construct $\Theta_p =$ analogue of D_g .

Thm: $((\Theta_p)^d)(\tilde{f})(z) \in R$

where z is an R -valued CM point

Furthermore, these values are the same up to some well-determined period